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The sheath at an electrode close to plasma potential

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Abstract. It is shown that a sheath with a monotonic potential profile can exist at all negative voltages on an electrode in a plasma. When the electrode is close to plasma potential it is no longer valid to ignore those electrons which reach the electrode. The velocities with which ions approach the sheath satisfy a 'sheath criterion', which depends upon the potential drop across the sheath and not just on the electron temperature in the plasma.

1. Introduction

In the usual analysis of a sheath covering a negatively biased electrode in a low-pressure plasma (Langmuir and Mott-Smith Jr. 1924, Chen 1965) it is assumed that electrons have a Boltzmann density distribution inside the sheath. This is a good approximation when the flux of electrons reaching the electrode is small compared with the random flux of electrons at the sheath edge. In this paper we consider a sheath at a negative electrode close to plasma potential. Here the assumption of a Boltzmann density distribution is not reasonable and it is necessary to distinguish between the electrons which are reflected in the sheath and those which are transmitted to the electrode. We also examine the behaviour of the so-called 'sheath criterion' (Bohm 1949) when the probe is close to plasma potential. This is a condition for the sheath potential to grow monotonically and relates the ion velocities at the sheath edge to the electron temperature in the plasma.

2. Charge density inside the sheath

Consider a plane electrode immersed in a plasma composed of positive ions and thermal electrons. When the voltage applied to the electrode is very negative with respect to the plasma potential, an ion-rich sheath develops over the surface of the electrode and most electrons are reflected inside the sheath. The ions may acquire energy from weak electric fields in a plasma region in front of the electrode, known as the 'pre-sheath', and when they arrive at the sheath edge they must satisfy the so-called sheath criterion

$$\tfrac{1}{2}M\langle v_i^{-2}\rangle^{-1} \geqslant \tfrac{1}{2}kT_e \tag{1}$$

where M is the ion mass, k is Boltzmann's constant and T_e is the electron temperature. The symbol $\langle\rangle$ denotes averaging over the component of ion velocity v_i normal to the plane of the electrode. The above form of the criterion was derived by Harrison and Thompson (1959) and is more general than that of Bohm—in which it was assumed that the ions incident at the sheath edge were monoenergetic. Ignoring collisions inside the sheath, all ions which arrive at the sheath edge are accelerated across the sheath and are collected by the electrode. Hence, when the electrode potential is very negative, the current drawn by the electrode is almost entirely due to ions.

As the voltage applied to the electrode is made less negative an increasing number of electrons have sufficient energy to reach the electrode. The floating potential V_f

is the potential at which the electron and ion currents are exactly equal, and is approximately given by

$$n_s e \langle v_i \rangle = \tfrac{1}{4} n_s e \left(\frac{8kT_e}{\pi m}\right)^{1/2} \exp\left(\frac{eV_f}{kT_e}\right) \tag{2}$$

where n_s is the ion/electron density at the sheath edge, e and m are the magnitude of the electron charge and mass respectively, and v_i refers to the ion velocity at the sheath edge. Rearranging equation (2) gives

$$V_f = \frac{kT_e}{e} \ln\left\{\left(\frac{2\pi m}{kT_e}\right)^{1/2} \langle v_i \rangle\right\}.$$

This shows that the floating potential is typically a few times the electron temperature (in volts). In obtaining the expression for the electron current density on the right-hand side of (2) it has been assumed that electrons have a Boltzmann density distribution inside the sheath, i.e.

$$n_e = n_s \exp\left(\frac{eV}{kT_e}\right) \tag{3}$$

where V is the electrostatic potential at any point. This assumption is reasonable when the flux of electrons transmitted across the sheath is small compared with the random flux of electrons at the sheath edge.

In this paper we are concerned in particular with the structure of the sheath when the potential at the electrode lies somewhere in the range between the floating potential and the plasma potential. In this range it is no longer valid to assume that the transmitted flux is negligible compared with the random flux, and the Boltzmann expression (3) for the electron density in the sheath is no longer a good approximation.

In general, we shall assume that the potential in the sheath is monotonic and that the electrons are in thermal equilibrium with a Maxwellian distribution of velocities at the sheath edge:

$$f_e = A \exp\left(-\frac{mv_e^2}{2kT_e}\right). \tag{4}$$

Some electrons have sufficient energy to reach the electrode but the rest are reflected inside the sheath. If V_s is the potential drop between the sheath edge and the electrode, then the first group comprises all those electrons which, at the sheath edge, are in the velocity range

$$\left(-\frac{2eV_s}{m}\right)^{1/2} \leqslant v_e \leqslant \infty.$$

Ignoring collisions, the transmitted electrons obey the conservation laws of mass and energy, and their density at any point inside the sheath is given by

$$\begin{aligned} n_{et} &= A \int_{(-2eV_s/m)^{1/2}}^{\infty} \left(1 + \frac{2eV}{mv_e^2}\right)^{-1/2} \exp\left(-\frac{mv_e^2}{2kT_e}\right) dv_e \\ &= \tfrac{1}{2} A \left(\frac{2\pi kT_e}{m}\right)^{1/2} \left[1 - \operatorname{erf}\left\{-\frac{e(V_s - V)}{kT_e}\right\}^{1/2}\right] \exp\left(\frac{eV}{kT_e}\right) \end{aligned} \tag{5}$$

where $\mathrm{erf}(y)$ is the error function

$$\mathrm{erf}(y) = \frac{2}{\pi^{1/2}} \int_0^y \exp(-z^2)\,\mathrm{d}z.$$

At any point inside the sheath the reflected electrons lie in the velocity range

$$-u \leqslant v_\mathrm{e} \leqslant u$$

where $u = \{2e(V_\mathrm{s} - V)/m\}^{1/2}$ and their density is given by

$$n_\mathrm{er} = A \int_{-u}^{u} \exp\left(-\frac{\frac{1}{2}mv_\mathrm{e}^2 - eV}{kT_\mathrm{e}}\right) \mathrm{d}v_\mathrm{e} = A\left(\frac{2\pi kT_\mathrm{e}}{m}\right)^{1/2} \mathrm{erf}\left\{-\frac{e(V_\mathrm{s} - V)}{kT_\mathrm{e}}\right\}^{1/2} \exp\left(\frac{eV}{kT_\mathrm{e}}\right). \tag{6}$$

Adding equations (5) and (6), and putting $n_\mathrm{e} = n_\mathrm{s}$ at the sheath edge, we have

$$A = 2n_\mathrm{s}\left(\frac{m}{2\pi kT_\mathrm{e}}\right)^{1/2} \Big/ \left\{1 + \mathrm{erf}\left(-\frac{eV_\mathrm{s}}{kT_\mathrm{e}}\right)^{1/2}\right\}. \tag{7}$$

It is convenient to define the following normalized variables:

$$\nu_\mathrm{e} = n_\mathrm{e}/n_\mathrm{s}, \qquad \eta = eV/kT_\mathrm{e}, \qquad \eta_\mathrm{s} = eV_\mathrm{s}/kT_\mathrm{e}. \tag{8}$$

The total normalized electron density becomes

$$\nu_\mathrm{e} = \frac{n_\mathrm{et} + n_\mathrm{er}}{n_\mathrm{s}}$$

or, using equations (5)–(7), we have

$$\nu_\mathrm{e} = \frac{1 + \mathrm{erf}\{-(\eta_\mathrm{s} - \eta)\}^{1/2}}{1 + \mathrm{erf}(-\eta_\mathrm{s})^{1/2}} \exp \eta. \tag{9}$$

Now consider the ions. We shall assume neutrality at the sheath edge and that there are no collisions or ionization inside the sheath. Applying the conservation laws of mass and energy, the ion density at any point inside the sheath is given by

$$n_\mathrm{i} = n_\mathrm{s} \int_0^\infty \left(1 - \frac{2eV}{Mv_\mathrm{i}^2}\right)^{-1/2} f_\mathrm{i}(v_\mathrm{i})\,\mathrm{d}v_\mathrm{i} \tag{10}$$

where $f_\mathrm{i}(v_\mathrm{i})$ is the ion velocity distribution function at the sheath edge. Setting

$$\nu_\mathrm{i} = n_\mathrm{i}/n_\mathrm{s}, \qquad \phi = Mv_\mathrm{i}^2/kT_\mathrm{e} \tag{11}$$

equation (10) becomes

$$\nu_\mathrm{i} = \left\langle\left(1 - \frac{2\eta}{\phi}\right)^{-1/2}\right\rangle. \tag{12}$$

Equations (9) and (12) are expressions for the (normalized) electron and ion densities as a function of (normalized) voltage η through the sheath.

3. The sheath criterion

Poisson's equation in one dimension is

$$\epsilon_0 \frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = -e(n_\mathrm{i} - n_\mathrm{e}).$$

Defining the non-dimensional length

$$X = x/\lambda_D \tag{13}$$

where

$$\lambda_D = \left(\frac{\epsilon_0 k T_e}{n_s e^2}\right)^{1/2} \tag{14}$$

is the Debye length at the sheath edge, Poisson's equation becomes

$$\frac{d^2\eta}{dX^2} = \nu_e - \nu_i. \tag{15}$$

Now from equations (9) and (12) we note that ν_e and ν_i are functions of η alone. Expanding (15) for small η inside the sheath using Taylor's theorem, we have

$$\frac{d^2\eta}{dX^2} = (\nu_e' - \nu_i')_s\,\eta + \tfrac{1}{2}(\nu_e'' - \nu_i'')_s\,\eta^2 + \ldots \tag{16}$$

where the prime denotes differentiation with respect to η. For sufficiently small η, the second term on the right-hand side of (16) is negligible compared with the first term, and (16) reduces to

$$\frac{d^2\eta}{dX^2} \simeq (\nu_e' - \nu_i')_s\,\eta.$$

Hence η decreases monotonically with X on entering the sheath only if

$$(\nu_e' - \nu_i')_s \geqslant 0. \tag{17}$$

Differentiating (9) and (12), setting $\eta = 0$ at the sheath edge and substituting in (17) yields

$$\langle \phi^{-1} \rangle^{-1} \geqslant \frac{1}{1 + F(\eta_s)} \tag{18}$$

where F is given by

$$F(\eta_s) = \frac{\exp \eta_s}{(-\pi\eta_s)^{1/2}\{1 + \mathrm{erf}(-\eta_s)^{1/2}\}}. \tag{19}$$

In terms of the unnormalized variables, (18) is just

$$\tfrac{1}{2}M\langle v_i^{-2}\rangle^{-1} \geqslant \frac{\tfrac{1}{2}kT_e}{1 + F(\eta_s)}. \tag{20}$$

Comparing inequalities (1) and (20), we see that the effect of allowing for the transmitted electrons in addition to the reflected electrons is to reduce the right-hand side of the inequality by a factor $1 + F(\eta_s)$. When η_s is large, we have $F(\eta_s) \simeq 0$ and the effect is negligible. Thus, when the electrode voltage is very negative the effect is small and previous analyses are substantially correct. On the other hand, as $\eta_s \to 0$ we have $F(\eta_s) \to \infty$ and the right-hand side of (20) tends to zero; in this case, when the electrode is at plasma potential, there is no restriction imposed on the ion velocity distribution function at the sheath edge and the necessity for a pre-sheath disappears.

In general, we find that the sheath criterion is actually dependent on the potential drop across the sheath. Figure 1 shows the sheath criterion as a function of the normalized sheath potential drop. The criterion is satisfied on or above the line given by the

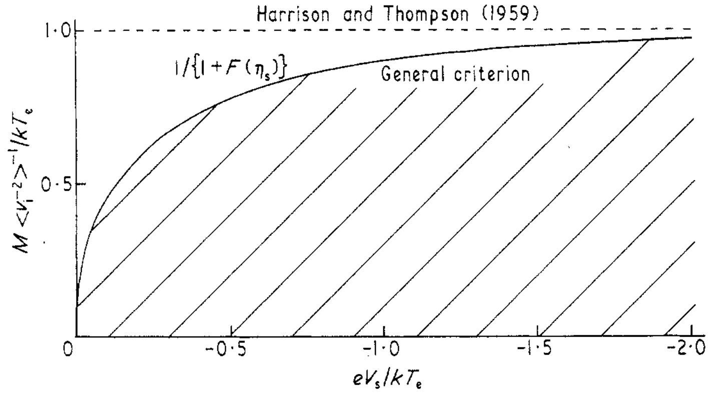

Figure 1. Variation of the sheath criterion with the normalized potential drop across the sheath.

equation

$$\frac{M\langle v_i^{-2}\rangle^{-1}}{kT_e} = \frac{1}{1+F(\eta_s)}. \tag{21}$$

In Harrison and Thompson's (1959) analysis, which did not take into account the transmitted electrons, the criterion was defined by the line

$$\frac{M\langle v_i^{-2}\rangle^{-1}}{kT_e} = 1.$$

This appears as a horizontal line in figure 1.

4. The potential profile

Multiplying both sides of equation (15) by $(\mathrm{d}\eta/\mathrm{d}X)\,\mathrm{d}X$ and integrating gives the normalized Maxwell stress at any point in the sheath as

$$\tfrac{1}{2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}X}\right)^2 = \int_0^\eta \{\nu_e(\xi)-\nu_i(\xi)\}\,\mathrm{d}\xi = G(\phi,\eta_s,\eta) \tag{22}$$

say; substituting for ν_e and ν_i from (9) and (12) respectively in (22) and integrating (setting $\mathrm{d}\eta/\mathrm{d}X = 0$ at the sheath edge), yields

$$G(\phi,\eta_s,\eta) = \left(\frac{2}{\pi^{1/2}}[(-\eta_s)^{1/2}-\{-(\eta_s-\eta)\}^{1/2}]\exp\eta_s-(1-\exp\eta)-\mathrm{erf}(-\eta_s)^{1/2}\right.$$
$$\left.+\mathrm{erf}\{-(\eta_s-\eta)\}^{1/2}\exp\eta\right)/\{1+\mathrm{erf}(-\eta_s)^{1/2}\}+\left\langle\phi\left\{\left(1-\frac{2\eta}{\phi}\right)^{1/2}-1\right\}\right\rangle. \tag{23}$$

Rearranging (22), and integrating from the electrode ($\eta = \eta_s$) to any point in the sheath, we obtain the potential profile as

$$X = \int_{\eta_s}^{\eta} \{2G(\phi,\eta_s,\xi)\}^{-1/2}\,\mathrm{d}\xi \tag{24}$$

where X is measured from the point where $\eta = \eta_s$. We take the origin to be at the electrode rather than the sheath edge, since the sheath edge is not well defined. $d\eta/dX \to 0$ asymptotically as $\eta \to 0$—a result which can be obtained by integrating (16) twice and rearranging, for small powers of η in the neighbourhood of the sheath edge. So far the analysis is quite general and applies for any reasonably well-behaved ion velocity distribution. In order to evaluate an actual potential profile numerically, the particular ion velocity distribution must be considered.

As an example we may suppose that the ions are monoenergetic—a case that is frequently considered in the literature.

5. Numerical solution for the case of monoenergetic ions

At the sheath edge we set $\phi = \phi_0$, corresponding to the lower bound of inequality (18):

$$\phi_0 = \frac{1}{1+F(\eta_s)} \tag{25}$$

i.e. the ions have the *minimum* kinetic energy needed to satisfy the sheath criterion.

The potential profiles for various values of the normalized potential drop across the sheath eV_s/kT_e are given in figures 2 and 3. In all cases the sheath is everywhere

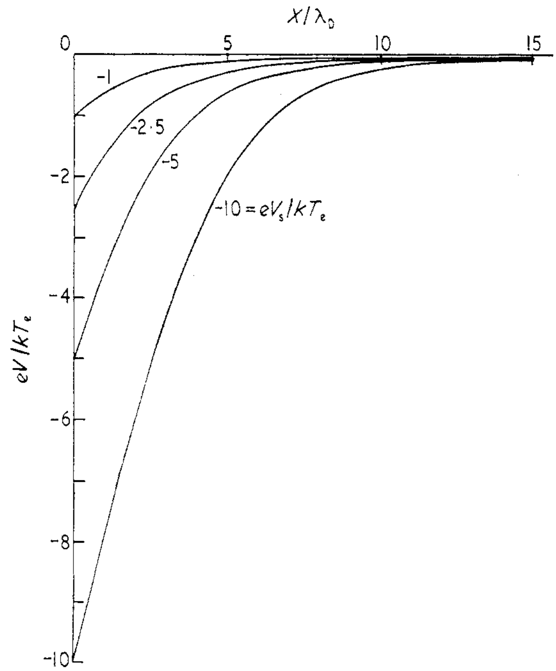

Figure 2. Normalized potential profiles in the sheath for the range $-10 \leqslant eV_s/kT_e \leqslant -1$.

ion-rich and a self-consistent monotonic potential profile is obtained. Hence, it appears that the concept of a sheath is meaningful even when the potential at the electrode is close to the plasma potential.

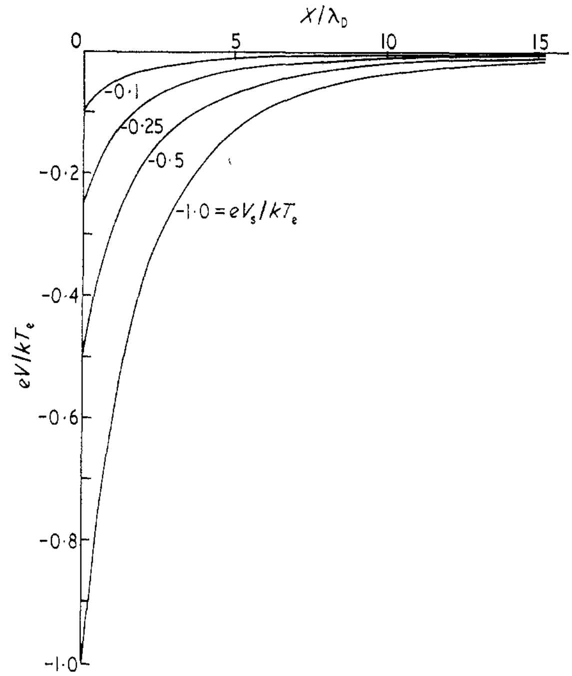

Figure 3. Normalized potential profiles in the sheath for the range $-1{\cdot}0 \leqslant eV_s/kT_e \leqslant -0{\cdot}1$.

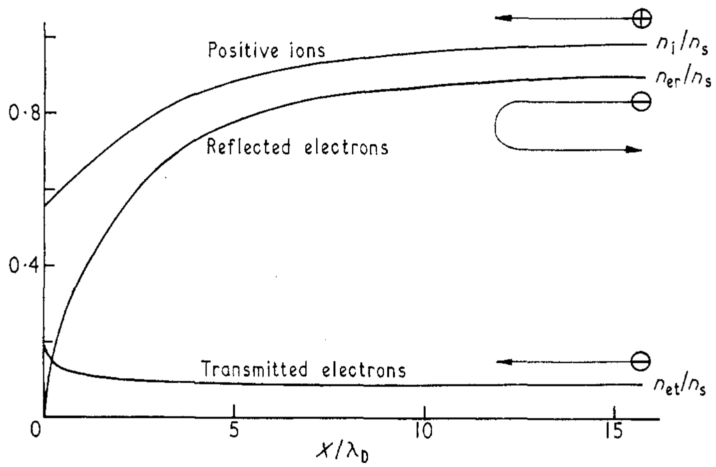

Figure 4. Normalized density profiles for $eV_s/kT_e = -1{\cdot}0$ ($\phi_0 = 0{\cdot}899$).

Figure 4 shows a typical set of normalized density profiles in the sheath for $eV_s = -1{\cdot}0$ ($\phi_0 = 0{\cdot}899$), the curve which appears in both figures 2 and 3. The ion density is attenuated as the ions are accelerated but the density of the transmitted group of electrons increases since they are retarded in the sheath. The density of the reflected group of electrons is attenuated through the sheath and falls to zero at the electrode.

6. Conclusions

A self-consistent model of a sheath at an electrode close to plasma potential can be obtained provided that the ions incident at the sheath edge satisfy a sheath criterion which is dependent on the voltage drop across the sheath.

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References

BOHM, D., 1949, *The Characteristics of Electrical Discharges in Magnetic Fields,* Eds A. Guthrie and R. K. Wakerling (New York: McGraw-Hill), chap. 3.

CHEN, F. F., 1965, *Plasma Diagnostic Techniques*, Eds R. H. Huddlestone and S. L. Leonard (London: Academic), chap. 4.

HARRISON, E. R., and THOMPSON, W. B., 1959, *Proc. Phys. Soc.* B, **74**, 145.

LANGMUIR, I., and MOTT-SMITH, H., Jr., 1924, *Gen. Elect. Rev.*, **27**, 449, 538, 616, 762.